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A novel piecewise multivariate function approximation method via universal matrix representation

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Abstract Multivariance in science and engineering causes problematic situations even for continous and discrete cases. One way to overcome this situation is to decrease the multivariance level of the problem by using a divide—and—conquer based method. In this sense, Enhanced Multivariance Product Representation (EMPR) plays a part in the considered scenario and acts successfully. This method brings up a finite expansion to represent a multivariate function in terms of less-variate functions with the assistance of univariate support functions. This work aims to propose a new EMPR based algorithm which has two new features that improves the determination process of each expansion component through Fluctuation Free Integration method, which is an efficient method in evaluating multiple integrals through a universal matrix representation, and increases the approximation quality through inserting a piecewise structure into the standard EMPR algorithm. This new method is called Fluctuation Free Integration based piecewise EMPR. Some numerical implementations are also given to examine the performance of this proposed method.

Keywords EMPR · Piecewise functions · Multidimensional problems · Approximation · Matrix representation · Numerical integration

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1 Introduction

Enhanced Multivariance Product Representation (EMPR) is a recently developed tool for representing multivariate functions [1–3]. It is an extension to high dimensional model representation (HDMR) which is used for approximating multivariate functions and modelling multivariate problems [4–9]. Both methods are based on divide-and-conquer philosophy and they include multivariate integration evaluations under a product type weight over orthogonal geometries. Actually, the EMPR method covers HDMR because if all support functions are taken as unit constant functions, then EMPR acts as HDMR. One of the most important difference between these two methods is that the HDMR method produces better approximations when the given multivariate function has purely additive nature whereas HDMR becomes poor if the given function has purely or dominantly multiplicative nature. This time, instead of HDMR, the EMPR method should be used for obtaining high quality approximations to the multivariate functions under consideration.

The EMPR expansion is a finite sum and has 2^N terms to exactly represent the given multivariate function having N independent variables. The first term of the expansion is the constant component, the next $N(N-1)/2$ terms are the univariate ones which are followed by the higher variate components. The main purpose in the EMPR method is to obtain the structure of each EMPR component. However, the algorithm in finding these components includes multiple integrations, such as to get the structure of the constant EMPR component, we need to evaluate N-tuple integrals while $N-1$-tuple integration evaluation is needed for the structure of each univariate EMPR component. The higher variate components need similar evaluations. This means, when N tends to grow unboundedly, both the total number of expansion terms to be specified and the total number of integrals to be evaluated for this specification process grow undesirably. But it is obvious that to calculate whole terms of that expansion is a quite difficult process. For these reasons, the EMPR method is assumed to be as an approximation method, that is, only first few terms of its expansion are utilized in the representation. Thus, the approximation quality becomes a very important issue for the method. This can be achieved by taking as many terms as possible from the expansion. However, it is not a preferred case since this increases the computational complexity dramatically. The other way is to specify the most appropriate support function or weight function structures. Thus, the optimization processes on either weight or support functions are needed. There is a study related with optimization on weight function [10] but optimization on support functions is quite hard because of the nonlinear equations encountered during the corresponding process.

In this work, we focus on two important points. One is to get rid of evaluating multiple integrals analytically. For this purpose, we insert a new procedure into EMPR philosophy to easily get the solutions of the multiple integrations appearing in the considered method. This procedure includes the Fluctuation Free Integration method. This method uses the Fluctuationlessness Theorem and allows us to use summations instead of integrals with the help of universal matrix representation of the independent variables of the function under consideration to evaluate the multiple integrals appearing in EMPR [11–14].

The second focus point of this work is to increase the quality of the mentioned EMPR approximations through a newly developed EMPR based method. This new method is called "Fluctuation Free Integration Based Piecewise EMPR (FFI-Piecewise EMPR)". In this new algorithm, we divide the whole N dimensional EMPR geometry corresponding to the related problem domain into a number of N dimensional subgeometries and then we obtain Fluctuation Free Integration Based EMPR (FFI-EMPR) approximants for each subgeometry. Finally, the superposition of these subgeometries are built to obtain an approximate analytical structure for the given problem domain.

Using subgeometries are also a way of constructing models occured in different kind of problems such as image restoration [15], periodic function approximation [16] and multivariate data modelling [17].

The paper is organized as follows. The second section includes the mathematical background related to the details of the proposed method. This mathematical background is composed of the standard EMPR method and the Fluctuation Free Integration method. The third section contains the preliminary steps of our proposed method. The algorithm of FFI-EMPR is given in this section. The FFI-Piecewise EMPR method, which is the final algorithm proposed in this work, is given in the fourth section. The fifth section covers a number of numerical implementations to observe the performance of the proposed method while concluding remarks are discussed in the last section.

2 Mathematical background

The EMPR method has a finite expansion including support functions to represent a given multivariate function in terms of less variate functions. It is an extension to HDMR philosophy and the proposed method of this work is based on this algorithm. This section covers the basic details of this already known method.

In addition, it is also well known that the evaluation of multiple integrals appearing in this algorithm urges us to develop a new procedure to get rid of the disadvantages of the mentioned integrals. This work uses the Fluctuation Free Integration method for this purpose. The details of this method are also given below.

2.1 The EMPR method

The EMPR expansion aims to decompose a multivariate function, $f(x_1, \ldots, x_N)$, having N independent variables, into less variate function components through the following finite expansion

$$f(x_1, \ldots, x_N) = f_0 \prod_{j=1}^{N} s_j(x_j) + \sum_{i=1}^{N} f_i(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{N} s_j(x_j) + \sum_{\substack{i_1, i_2 = 1 \\ i_1 < i_2}}^{N} f_{i_1 i_2}(x_{i_1}, x_{i_2}) \prod_{\substack{j=1 \\ j \neq i_1, i_2}}^{N} s_j(x_j) + \cdots + f_{1\ldots N}(x_1, \ldots, x_N) \tag{1}$$

where $s_j(x_j)$s are the support functions. Here, the problem domain is defined over the hyperprism $[a_1, b_1] \times \cdots \times [a_N, b_N]$ for generality [1,18]. The method does not bring any restrictions on the structure of the support functions, that is, there is no obligation that each support function should be univariate.

The EMPR components can be uniquely determined by using any chosen support function set. Of course, the selection process of the support functions is a very important issue for determining the EMPR components accurately, that is, the structure of the support functions effect the performance of EMPR directly. Hence, the approximation quality mostly depends on the selection of the support functions appropriately. Constructing a new method for support function selection is out of scope of this work and in this sense the following definition will be used to determine the efficient support function structures [1]

$$s_j(x_j) = \frac{\int_{a_1}^{b_1} dx_1 \cdots \int_{a_{j-1}}^{b_{j-1}} dx_{j-1} \int_{a_{j+1}}^{b_{j+1}} dx_{j+1} \cdots \int_{a_N}^{b_N} dx_N f(x_1, \ldots, x_N) \prod_{\substack{i=1 \\ i \neq j}}^{N} W_i(x_i)}{\left[\int_{a_j}^{b_j} dx_j W_j(x_j) \left[\int_{a_1}^{b_1} dx_1 \cdots \int_{a_{j-1}}^{b_{j-1}} dx_{j-1} \int_{a_{j+1}}^{b_{j+1}} dx_{j+1} \cdots \int_{a_N}^{b_N} dx_N f(x_1, \ldots, x_N) \right]^2 \prod_{\substack{i=1 \\ i \neq j}}^{N} W_i(x_i) \right]^{\frac{1}{2}}} \tag{2}$$

Now, to determine the components of the EMPR expansion given in (1), we define the following vanishing conditions [1]

$$\int_{a_{i_\ell}}^{b_{i_\ell}} dx_{i_\ell} W_{i_\ell}\left(x_{i_\ell}\right) s_{i_\ell}\left(x_{i_\ell}\right) f_{i_1,\ldots,i_k}\left(x_{i_1}, \ldots, x_{i_k}\right) = 0, \quad x_{i_\ell} \in \left\{x_{i_1}, \ldots, x_{i_k}\right\} \tag{3}$$

The normalization conditions given below are also defined to get rid of the complicated formulae similar with the HDMR philosophy [1]

$$\int_{a_i}^{b_i} dx_i \, W_i(x_i) = 1, \quad \int_{a_i}^{b_i} dx_i \, W_i(x_i) s_i\,(x_i)^2 = 1, \quad 1 \leq i \leq N \tag{4}$$

We also define the following projection operators to determine the constant and univariate terms of EMPR

$$\mathscr{I}_0 f(x_1, \ldots, x_N) \equiv \int_{a_1}^{b_1} dx_1 W_1(x_1) \cdots \int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{j=1}^{N} s_j(x_j) f(x_1, \ldots, x_N) \tag{5}$$

$$\mathscr{I}_i f(x_1, \ldots, x_N) \equiv \int_{a_1}^{b_1} dx_1 W_1\,(x_1) \cdots \int_{a_{i-1}}^{b_{i-1}} dx_{i-1} W_{i-1}\,(x_{i-1}) \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} W_{i+1}$$

$$\times (x_{i+1}) \cdots \int_{a_N}^{b_N} dx_N W_N (x_N) \prod_{\substack{j=1 \\ j \neq i}}^{N} s_j \left(x_j\right) f (x_1, \ldots, x_N) \quad (6)$$

If the operator, $\mathscr{I}_0$, is applied to the both sides of the EMPR expansion given in (1) under the assumptions given in (3) and (4), the general structure of the constant term is obtained as

$$f_0 = \int_{a_1}^{b_1} dx_1 W_1(x_1) \cdots \int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{j=1}^{N} s_j(x_j) f(x_1, \ldots, x_N) \quad (7)$$

while the structure of the univariate terms is determined through the operator $\mathscr{I}_i$ as follows

$$f_i(x_i) = \int_{a_1}^{b_1} dx_1 W_1 (x_1) \cdots \int_{a_{i-1}}^{b_{i-1}} dx_{i-1} W_{i-1} (x_{i-1}) \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} W_{i+1} (x_{i+1}) \cdots$$
$$\times \int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{\substack{j=1 \\ j \neq i}}^{N} s_j \left(x_j\right) f (x_1, \ldots, x_N) - f_0 s_i (x_i) \quad (8)$$

Because of taking all EMPR components into consideration causes high computational complexity, the EMPR expansion is truncated at some level, that is, only a few first components are used to obtain a representation for the given multivariate function. This is why we are going to deal with at most the univariate EMPR components in this work. In this sense, we do not intend to give more details about the general structures of the higher variate terms of EMPR. However, they can be determined by using the same philosophy.

Truncating the EMPR expansion at some level results in representing the given multivariate function through EMPR approximants. The constant and univariate EMPR approximants can be expressed as

$$\pi_0(x_1, \ldots, x_N) = f_0 \prod_{j=1}^{N} s_j(x_j),$$
$$\pi_1(x_1, \ldots, x_N) = \pi_0(x_1, \ldots, x_N) + \sum_{i=1}^{N} f_i(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{N} s_j(x_j) \quad (9)$$

while the approximants including higher variate EMPR components can be written in a similar manner.

To measure the EMPR approximation qualities, the following EMPR truncation quality measurers are utilized for the constant and univariate cases [1]

$$\sigma_0^{(e)} \equiv \frac{1}{\|f\|^2}\left\|f_0\prod_{j=1}^{N}s_j\right\|^2,$$
$$\sigma_1^{(e)} \equiv \frac{1}{\|f\|^2}\sum_{i=1}^{N}\left\|f_i\prod_{\substack{j=1\\ j\neq i}}^{N}s_j\right\|^2+\sigma_0^{(e)} \tag{10}$$

These measurers satisfy the non-decreasing rule, that is, $0 \leq \sigma_0^{(e)} \leq \sigma_1^{(e)} \cdots \leq \sigma_N^{(e)} = 1$. This rule allows us to use these measurers as an error analysis tool for the EMPR approximations.

2.2 The Fluctuation Free Integration method

The fluctuation free integration method uses the Fluctuationlessness Theorem to evaluate multiple integrals numerically. The Fluctuationlessness theorem which was proven by Professor Demiralp has gone from strength to strength by its utilizations in several fields.This theorem has been applied to the problems of mathematics, mechanics and chemistry like numerical integrations, function approximations and quantum optimal control problems [11–14]. Depicted theorem dictates an equality between two infinite dimensional matrices. The first one is the matrix representation of an operator standing in infinite dimensional Hilbert space, $\mathscr{H}$, which multiplies its operand by an analytic, square integrable univariate function, $f(x)$. Thus, $f(x)$ also resides in $\mathscr{H}$. The second matrix is the image of the matrix representation that stands for the algebraic multiplication operator which multiplies its operand by the independent variable x, under the f function. The descripted equality, of course, turns into an approximation if the corresponding matrices are restricted to a finite dimensional square matrices. If the first and the second operators are notated as $\widehat{f}$ and $\widehat{x}$, respectively, the following approximation can be written

$$\mathscr{M}_{\widehat{f}}^{(n)} \approx f\left(\mathscr{M}_{\widehat{x}}^{(n)}\right) \tag{11}$$

where $\mathscr{M}_{\widehat{f}}^{(n)}$ takes place for the truncated matrix representation of the operator $\widehat{f}$ and $\mathscr{M}_{\widehat{x}}^{(n)}$ stands for the matrix representation of $\widehat{x}$. Here, the superscript (n), which is a positive integer, denotes the truncation level.

Momentarily, we should place the matrix representation of $\widehat{f}$ on one side and pay our attention to the other one, $\mathscr{M}_{\widehat{x}}^{(n)}$. To construct this matrix, a basis set, constituted by linearly independent functions, has to be considered. It is not needed that the elements of this basis set are orthonormal to each other. If they are not orthonormal, they can be succeeded by using Gram–Schmidt orthonormalization process or Cholesky

method. Although any basis set possessing the mentioned features can be utilized, the one including the powers of the independent variable x will be employed for universality which is an important property to obtain a general algorithm. Thus, the unorthonormalized basis set can be expressed as follows

$$\mathscr{V}_n = \left\{x^{i-1}\right\}_{i=1}^{n} \tag{12}$$

If $\mathscr{V}_n$ is orthonormalized then

$$\mathscr{U}_n = \{u_i\}_{i=1}^{n} \tag{13}$$

is obtained. Here, u_is are the orthonormalized elements evaluated by using the following inner product

$$(g,h) = \int_0^1 dx\, w(x) g(x) h(x), \quad g,h \in \mathscr{H} \tag{14}$$

During the orthonormalization procedure, the weight function, $w(x)$, will be taken as unit constant function in our further analysis for simplicity. The reason for the usage of the unit interval in the inner product is again the universality as one can easily confirm. Using this inner product defined on the relevant interval, u_is are computed as the *Shifted Legendre Polynomials* with the coefficients $\sqrt{2i-1}$. Hereby, an orthonormal basis set with respect to the inner product given in (14) on the interval [0, 1] is attained.

The entries of the matrix $\mathscr{M}_{\widehat{x}}^{(n)}$ can be calculated under favour of the inner product given in (14) and the elements given in (13) as

$$\mathbf{e}_i^{(n)^T} \mathscr{M}_{\widehat{x}}^{(n)} \mathbf{e}_j^{(n)} = \left(u_i, \widehat{x} u_j\right) = \int_0^1 dx\, u_i(x) x u_j(x), \quad i,j = 1,\ldots,n \tag{15}$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the unit Cartesian vectors which take zeros as their entries except that the ith and the jth ones, respectively. The matrix whose ith row jth column element given above is also called as "Universal Matrix". On the other hand, entries of the matrix representation of the $\widehat{f}$ operator can be calculated in a similar way

$$\mathbf{e}_i^{(n)^T} \mathscr{M}_{\widehat{f}}^{(n)} \mathbf{e}_j^{(n)} = \left(u_i, \widehat{f} u_j\right) = \int_0^1 dx\, u_i(x) f(x) u_j(x), \quad i,j = 1,\ldots,n \tag{16}$$

Since $\mathscr{M}_{\widehat{x}}^{(n)}$ is symmetric and positive definite, all of its eigenvalues are real and reside in the interval [0, 1] with no multiplicity. Also, the left and the right eigenvectors related to a certain eigenvalue are equal. With the help of these facts, $\mathscr{M}_{\widehat{x}}^{(n)}$ can be expressed

as a linear combination including the outer products of its eigenvectors as

$$\mathscr{M}_{\widehat{x}}^{(n)} = \sum_{k=1}^{n} \lambda_k \boldsymbol{\xi}_k^{(n)} \boldsymbol{\xi}_k^{(n)^T} \tag{17}$$

where λ_ks are the eigenvalues and $\boldsymbol{\xi}_k$s are the related eigenvectors. This expansion is known as the *spectral decomposition* of the symmetric matrix, $\mathscr{M}_{\widehat{x}}^{(n)}$.

Using the Fluctuationlessness Theorem given in (11) and the expansion given in (17) by combining the definitions (15) and (16) and also considering the reality that the very first element of the orthonormal basis set given in (13), u_1, is 1, the following result is obtained for the integral of the function, $f(x)$, over the unit interval

$$\begin{aligned}\int_0^1 dx f(x) &= \int_0^1 dx u_1(x) f(x) u_1(x) = \left(u_1, \widehat{f} u_1\right) \\ &= \mathbf{e}_1^{(n)^T} \mathscr{M}_{\widehat{f}}^{(n)} \mathbf{e}_1^{(n)} \approx \mathbf{e}_1^{(n)^T} f\left(\mathscr{M}_{\widehat{x}}^{(n)}\right) \mathbf{e}_1^{(n)} \\ &= \mathbf{e}_1^{(n)^T} \left(\sum_{k=1}^{n} f\left(\lambda_k\right) \boldsymbol{\xi}_k^{(n)} \boldsymbol{\xi}_k^{(n)^T}\right) \mathbf{e}_1^{(n)} = \sum_{k=1}^{n} f(\lambda_k) \left(\mathbf{e}_1^{(n)^T} \boldsymbol{\xi}_k^{(n)}\right)^2 \end{aligned} \tag{18}$$

Finally, the given univariate integral is rewritten in terms of summations. This relation is the resulting step of the Fluctuation Free Integration method [19]. To cope with the multiple integrals of EMPR, the Fluctuation Free Integration method can be utilized.

3 Fluctuation Free Integration based EMPR (FFI-EMPR)

As we have mentioned before, the EMPR method has a serious disadvantage due to so many multiple integral evaluations. To get rid of this disadvantage, the Fluctuation Free Integration method can be used. In this work, we aim to apply the properties of this method to EMPR to construct an efficient method based on piecewise philosophy that can be used in representing multivariate functions.

To construct the FFI-EMPR method, we need to use the relations given in (7) and (8) obtained for constant and univariate EMPR components.

The first step is to obtain the structure of the constant EMPR component by taking the features of the Fluctuation Free Integration method into consideration. Here, we use unit interval and unit weight for a simple explanation at the beginning. This gives us the following relation for the constant component

$$f_0 = \int_0^1 dx_1 s_1(x_1) \cdots \int_0^1 dx_N s_N(x_N) f(x_1, \ldots, x_N) \tag{19}$$

Here, there are N-tuple integrals and we calculate these integrals one by one in order by using Fluctuation Free Integration method. For that reason, we have to construct

the matrix representation of $\widehat{x}_i$ where $1 \leq i \leq N$. These matrix representations are $\mathscr{M}_{\widehat{x}_1}^{(n_1)}, \mathscr{M}_{\widehat{x}_2}^{(n_2)}, \ldots, \mathscr{M}_{\widehat{x}_N}^{(n_N)}$. Because we do not loose anything from the generality, we can also use the same single universal matrix instead of all these seperate ones. In this sense, we use only a single universal matrix with n dimension while some studies include N number of universal matrices having different dimensions [19]. We denote this matrix by $\mathscr{M}_{\widehat{x}}^{(n)}$. Using a single matrix representation also avoids high computational complexity. After the universal matrix construction, as stated in Sect. 2.2, we have to find the eigenvalues and the corresponding eigenvectors of $\mathscr{M}_{\widehat{x}}^{(n)}$.

Now we can start to evaluate N-tuple integrals beginning from the last one by using the relation (18). The following relation is given to express how the process proceeds

$$f_0 \approx \int_0^1 dx_1 s_1(x_1) \cdots \int_0^1 dx_{N-1} s_{N-1}(x_{N-1}) \mathbf{e}_1^{(n)T} \sum_{k_N=1}^{n} \left[f\left(x_1, \ldots, x_{N-1}, \lambda_N^{(k_N)}\right) \times s_N(\lambda_N^{(k_N)}) \boldsymbol{\xi}_{k_N}^{(n)} \boldsymbol{\xi}_{k_N}^{(n)T} \right] \mathbf{e}_1^{(n)} \tag{20}$$

where $\lambda_N^{(k_N)}$ stands for the k_N-th eigenvalue of the $n \times n$ dimensional matrix $\mathscr{M}_{\widehat{x}}^{(n)}$ and $\boldsymbol{\xi}_{k_N}^{(N)}$ is the corresponding eigenvector. If the above equation is reorganized, then the following expression is obtained.

$$f_0 \approx \sum_{k=1}^{n} \left(\mathbf{e}_1^{(n)T} \boldsymbol{\xi}_{k_N}^{(n)} \right)^2 \int_0^1 dx_1 s_1(x_1) \cdots \int_0^1 dx_{N-1} s_{N-1}(x_{N-1}) \times f\left(x_1, \ldots, x_{N-1}, \lambda_N^{(k_N)}\right) s_N(\lambda_N^{(k_N)}) \tag{21}$$

If we keep on evaluating the remaining integrals then we obtain the constant component of EMPR as follows.

$$f_0 \approx \sum_{k_1=1}^{n} \cdots \sum_{k_N=1}^{n} \left[\prod_{i=1}^{N} \left(\mathbf{e}_1^{(n)T} \boldsymbol{\xi}_{k_i}^{(n)} \right)^2 \right] f\left(\lambda_1^{(k_1)}, \ldots, \lambda_N^{(k_N)}\right) \prod_{i=1}^{N} s_i(\lambda_i^{(k_i)}) \tag{22}$$

After the determination process of the constant component, we can proceed to obtain the general structure of univariate EMPR components. For this purpose, we use same philosophy, that is, we evaluate the integral by starting from the last one and keep on to evaluate remaining integrals except the ith one as follows.

$$f_{i_1}(x_{i_1}) \approx \sum_{k_1=1}^{n} \cdots \sum_{k_{i_1-1}=1}^{n} \sum_{k_{i_1+1}=1}^{n} \cdots \sum_{k_N=1}^{n} \left[\prod_{\substack{m=1 \\ m\neq i_1}}^{N} \left(\mathbf{e}_1^{(n)T} \boldsymbol{\xi}_{k_m}^{(n)} \right)^2 \right] \times f\left(\lambda_1^{(k_1)}, \ldots, \lambda_{i_1-1}^{(k_{i_1-1})}, x_{i_1}, \lambda_{i_1+1}^{(k_{i_1+1})}, \ldots, \lambda_N^{(k_N)} \right) \prod_{\substack{m=1 \\ m\neq i_1}}^{N} s_m(\lambda_m^{(k_m)}) - f_0 \tag{23}$$

The constant and univariate components given in (22) and (23) are used to obtain the constant and univariate FFI-EMPR approximants by using the relations given in (9).

4 The Fluctuation Free Integration based piecewise EMPR method

The main idea in this new EMPR based method is to split the whole geometry into two or more subgeometries and to calculate the FFI-EMPR approximant for each subgeometry. This means that the new method produces an analytical structure in each subgeometry and it stands for a piecewise structure. To this end, the steps of FFI-Piecewise EMPR required to get the approximation are given as follows:

– Specify the total number of subintervals. If each independent variable, x_i resides in the interval $[a_i, b_i]$ where $1 \leq i \leq N$, we split each interval of the related independent variable into subintervals.

$$x_i^{(1)} \in \left[c_i^{(1)}, c_i^{(2)} \right], \quad x_i^{(2)} \in \left[c_i^{(2)}, c_i^{(3)} \right], \quad \ldots, \quad x_i^{(n_i)} \in \left[c_i^{(n_i)}, c_i^{(n_i+1)} \right], \quad c_i^{(1)} \equiv a_i, \quad c_i^{(n_i+1)} \equiv b_i, \quad 1 \leq i \leq N \tag{24}$$

That is, we split the interval $[a_1, b_1]$ into n_1 subintervals, the interval $[a_2, b_2]$ into n_2 subintervals and we keep on splitting the remaining intervals similar to these.

– Create the subgeometries. The total number of these subgeometries must be $p = n_1 \times n_2 \times \cdots \times n_N$.

$$\mathscr{D}^{(k)} \equiv x_1^{(j_1)} \times x_2^{(j_2)} \times \cdots \times x_N^{(j_N)}, \quad 1 \leq k \leq p, \quad 1 \leq j_i \leq n_i, \quad 1 \leq i \leq N \tag{25}$$

Although it is not an obligation, all domains for each independent variable are splitted uniformly in this work.

– Construct all normalized support functions by using the formula given in (2).
– Calculate the constant FFI-EMPR component for each subgeometry by using the equation given in (22).
– Obtain the constant FFI-EMPR approximant for each subgeometry. To evaluate these approximants, each constant FFI-EMPR component should be multiplied by

all corresponding support functions.

$$\pi_0(x_1,\ldots,x_N)^{(k)} = f_0^{(k)} \prod_{j=1}^{N} s_j(x_j), \quad 1 \le k \le p \tag{26}$$

Thus, a piecewise structure is obtained to make an approximation for a multivariate function under consideration. This approximant is called "the constant FFI-Piecewise EMPR approximant".

$$\pi_0(x_1,\ldots,x_N) = \begin{cases} f_0^{(1)} \prod_{j=1}^{N} s_j(x_j), & x_1,x_2,\ldots,x_N \in \mathscr{D}^{(1)} \\ f_0^{(2)} \prod_{j=1}^{N} s_j(x_j), & x_1,x_2,\ldots,x_N \in \mathscr{D}^{(2)} \\ \vdots & \\ f_0^{(p)} \prod_{j=1}^{N} s_j(x_j), & x_1,x_2,\ldots,x_N \in \mathscr{D}^{(p)} \end{cases} \tag{27}$$

- Evaluate the relative error values for each constant approximant of each subgeometry.
- Calculate the average of all relative error values to dictate the efficiency on the whole geometry which is $\mathscr{N}_{\pi_0}$.
- Determine the univariate FFI-EMPR component by using (23) and univariate approximation for each subgeometry. Similar with the constant FFI-Piecewise EMPR approximant, we can write the following piecewise structure

$$\pi_1(x_1,\ldots,x_N) = \begin{cases} f_0^{(1)} \prod_{j=1}^{N} s_j(x_j) + \sum_{i=1}^{N} f_i^{(1)}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} s_j(x_j), & x_1,\ldots,x_N \in \mathscr{D}^{(1)} \\ f_0^{(2)} \prod_{j=1}^{N} s_j(x_j) + \sum_{i=1}^{N} f_i^{(2)}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} s_j(x_j), & x_1,\ldots,x_N \in \mathscr{D}^{(2)} \\ \vdots & \\ f_0^{(p)} \prod_{j=1}^{N} s_j(x_j) + \sum_{i=1}^{N} f_i^{(p)}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} s_j(x_j), & x_1,\ldots,x_N \in \mathscr{D}^{(p)} \end{cases} \tag{28}$$

- Evaluate the relative error values for each univariate approximant of each subgeometry.
- Calculate the average of all relative error values to dictate the efficiency on the whole geometry which is $\mathscr{N}_{\pi_1}$.

The constant and univariate FFI-Piecewise EMPR approximants given in (27) and (28) now can be used to represent a given multivariate function through a piecewise structure. The argument of this work is to show that this newly developed method works better that standard EMPR for many cases. It is also expected that even the constant FFI-Piecewise EMPR approximant works better than the univariate EMPR approximant for the same case. To this end, the next section includes some numerical examples to examine the performance of FFI-Piecewise EMPR as well as to prove the mentioned arguments.

5 Numerical implementations

In this section, some certain numerical implementations are given to show the efficiency of our proposed method with the help of some tables and figures. The performance examination of the methods are done through the relative error values that are computed by dividing the norm of the difference between the obtained approximant and the original function by the norm of the original function itself. This means that the value most closest to 0 expresses the best approximation quality. Thus, the relative error values computed will be emphasized in the tables and the obtained approximants are going to be compared with the original multivariate function under consideration in the figures. All numerical results obtained in this section were computed by using MATLAB R2010a under the Linux Ubuntu 10.04 LTS operating system within 20 digits precision.

Since our new method includes the features of Fluctuation Free Integration method in its structure, we need to construct the matrix representation of the independent variables of the given problem. That is, we need the matrix, $\mathscr{M}_{\widehat{x}}^{(n)}$. Here, n is the dimension of this matrix and in this work we take n as 3. It is well known from the numerical experiments related with the Fluctuationlessness Approximation Theory that the results are satisfactory even the dimension of this universal matrix is very small numbers such as 3 or 4.

The testing multivariate functions are chosen by considering that they should have different functional structures to allow us to analyze the efficiency of the FFI-Piecewise EMPR method for different scenarios. These functions, namely from f_1 to f_7, are selected from 5 dimensional functional space, each within unit interval and are as follows

$$
\begin{aligned}
f_1(x_1,\ldots,x_5) &= \sum_{i=1}^{5} x_i, & f_2(x_1,\ldots,x_5) &= \left(\sum_{i=1}^{5} x_i\right)^5,\\
f_3(x_1,\ldots,x_5) &= \left(\sum_{i=1}^{5} x_i\right)^7, & f_4(x_1,\ldots,x_5) &= \prod_{i=1}^{5} x_i,\\
f_5(x_1,\ldots,x_5) &= \cos\left(\pi \sum_{i=1}^{5} x_i\right), & f_6(x_1,\ldots,x_5) &= \sum_{i=1}^{5} e^{i x_i},\\
f_7(x_1,\ldots,x_5) &= \frac{x_1^2+4x_2^2+9x_3^2}{1+9x_3^2+16x_4^2+25x_5^2}
\end{aligned}
\tag{29}
$$

where the first one is the summation of 5 independent variables. This function has a purely additive nature. The second multivariate function has monotonously increasing structure while propagating on all dimensions in the forward direction. For the third multivariate function, f_3, it is possible to revive all the comments about f_2. In addition, the increment of this function is sharper than the previous one. Thus, one may say that the curvilinearity of f_3 is greater than the function f_2 since all of its terms has

Table 1 Relative error values obtained through constant approximants

	$\mathcal{N}_{p_0}$	$\mathcal{N}_{\pi_0}^{(2)}$	$\mathcal{N}_{\pi_0}^{(4)}$	$\mathcal{N}_{\pi_0}^{(8)}$	$\mathcal{N}_{\pi_0}^{(12)}$
f_1	0.04106	0.02286	0.01206	0.00619	0.00416
f_2	0.11932	0.07064	0.03844	0.02003	0.01353
f_3	0.13088	0.08046	0.04459	0.02343	0.01587
f_4	0.00000	0.00000	0.00000	0.00000	0.00000
f_5	0.97170	0.82935	0.42842	0.21182	0.14078
f_6	0.14180	0.04587	0.01229	0.00312	0.00008
f_7	0.47297	0.23707	0.11423	0.05581	0.01561

the order 7 which is different from 5. In contrast to f_1, the fourth function is chosen as purely multiplicative to analyze the values of the relative errors while the structure of the considered multivariate function varies between additivity and multiplicativity. The fifth function is an oscillatory function since it includes the cosine function. The functions which have oscillatory structures are always hard to cope with for numerical approximation and integration methods. As a sixth example, the multivariate function, which is the summation of univariate exponential functions having different curvature parameters, is selected to investigate the case for the exponential structures. Finally, f_7 is chosen a bit different from the previous ones. This function is selected as a rational function which has no singularity point over the 5 dimensional real space but may have some over the complex space because of the polynomial structure appearing in the denominator.

Tables 1 and 2 include the relative error values obtained through the constant and univariate EMPR and FFI-Piecewise EMPR approximants respectively. In Table 1, $\mathcal{N}_{p_0}$ stands for the relative error value evaluated for the constant approximation obtained through the standard EMPR method while $\mathcal{N}_{\pi_0}^{(k)}$s stand for the relative error values of the constant FFI-Piecewise EMPR approximant where k lies between 2 and 12. That is, the implementations have uniformly divided 2, 4, 8 and 12 subintervals to be used in modelling the constructed problems through the testing functions given in (29).

Table 2 has the same characteristics in terms of total number of subintervals while $\mathcal{N}_{p_1}$ and $\mathcal{N}_{\pi_1}^{(k)}$s stand for the univariate standard EMPR and FFI-Piecewise EMPR approximants respectively.

The results given in both Tables 1 and 2 show us that the FFI-Piecewise EMPR method works better than the standard EMPR method for all testing functions. In addition, it can also be observed that the increment in the number of subintervals affects the approximation quality well and rises the efficiency up. Doubling the number of the subintervals halves the value of the corresponding relative error. This fact is one of our aims which we want to emphasize in this work. The results also show that the FFI-Piecewise EMPR method is still efficient even the sharpness of the function increases. One can say that even using 4 subintervals in modelling, very succesfull results are obtained for the most of the given testing functions. The proposed method can also handle the rational structure, that is, f_7. The FFI-Piecewise EMPR method can easily

Table 2 Relative error values obtained through univariate approximants

	$\mathcal{N}_{p_1}$	$\mathcal{N}_{\pi_1}^{(2)}$	$\mathcal{N}_{\pi_1}^{(4)}$	$\mathcal{N}_{\pi_1}^{(8)}$	$\mathcal{N}_{\pi_1}^{(12)}$
f_1	0.03898	0.02181	0.01154	0.00593	0.00399
f_2	0.06880	0.04408	0.02504	0.01332	0.00906
f_3	0.06301	0.04356	0.02581	0.01401	0.00959
f_4	0.00000	0.00000	0.00000	0.00000	0.00000
f_5	0.97170	0.51205	0.27443	0.13601	0.09042
f_6	0.01139	0.00293	0.00074	0.00018	0.00008
f_7	0.28065	0.10424	0.04468	0.02119	0.01387

Table 3 Standart deviation of relative error values obtained in the related subintervals for each constant FFI-Piecewise EMPR approximant

	#2	#4	#8	#12
f_1	0.02573	0.01940	0.01417	0.01170
f_2	0.06885	0.05445	0.04072	0.03387
f_3	0.07130	0.05844	0.04445	0.03717
f_4	0.00000	0.00000	0.00000	0.00000
f_5	0.00000	0.27707	0.25509	0.22322
f_6	0.01040	0.00312	0.00078	0.00001
f_7	0.28476	0.19483	0.13530	0.03674

overcome the disadvantages occured by rationality and decreases the relative error values drastically while the number of subintervals is increasing. It is obvious that the standard EMPR is very far away from expressing this function accurately.

When we compare the relative error results obtained through the constant and univariate FFI-Piecewise EMPR approximants, of course, the results obtained through the univariate approximations are better than the ones obtained by the constant approximations. However, the relative error values obtained through constant approximations in all testing functions are quite acceptable as the representation procedure. Hence, to reduce the mathematical and computational complexity of the method, even constant FFI-Piecewise EMPR approximants with 4 subintervals can be used instead of univariate approximations for the representation of the considered multivariate function. To this end, this work shows that we have two alternatives through FFI-Piecewise EMPR philosophy. One is using constant approximations to reduce the mentioned complexities within acceptable errors while the other is representing the multivariate function under consideration through the univariate approximants with better quality but higher complexity.

One additional examination about the performance of our new method is to observe the stability of the method in the subintervals used to construct the piecewise structure. For this purpose, after evaluating the relative error value for each subgeometry, we calculate the standard deviation of the obtained relative errors for the approximant

Table 4 Standart deviation of relative error values obtained in the related subintervals for each constant FFI-Piecewise EMPR approximant

	#2	#4	#8	#12
f_1	0.02428	0.01836	0.01343	0.01110
f_2	0.03497	0.02988	0.02323	0.01957
f_3	0.02750	0.02600	0.02119	0.01812
f_4	0.00000	0.00000	0.00000	0.00000
f_5	0.00000	0.10680	0.12455	0.11422
f_6	0.00000	0.00000	0.00000	0.00000
f_7	0.09567	0.05470	0.03791	0.03110

which represents the considered testing function. That is, when we have, for example, 4 relative error values for the FFI-Piecewise EMPR approximant with 4 subgeometries, we evaluate the standard deviation of these results to check out the stability of the method in terms of performance quality. The results are given in Tables 3 and 4 for the constant and univariate FFI-Piecewise EMPR approximations respectively. When we examine the tables, it is clear that almost all the standard deviation values are very close to 0 which means that our method works very well in all subgeometries while producing a piecewise structure for the considered testing function. The only problematic case is for the cosine function which has an oscillatory structure. This is an expected case and since for representing the whole geometry through FFI-Piecewise EMPR is successfull, this problematic case can be ignored.

Besides all these, we have two bivariate functions to analyze. We choose the following bivariate functions since we desire to deal with their surface plots.

$$f_8(x_1, x_2) = (x_1 + x_2)^4, \qquad f_9(x_1, x_2) = \frac{10 \sin\left(\sqrt{x_1^2 + x_2^2}\right)}{\sqrt{1 + x_1^2 + x_2^2}} \tag{30}$$

The first of these two testing functions is a multinomial. The other one has a little complicated structure involving sine, square root and fraction. The figures are constructed by plotting exact values of the corresponding function over the unit interval, [0, 1], and the constant approximants obtained utilizing standard EMPR and FFI-Piecewise EMPR with 4 subintervals. The colors used in figures for the exact function, standard EMPR and FFI-Piecewise EMPR approximants are red, blue and green respectively. In Fig. 1, it can be seen that the constant standard EMPR approximant represents the considered function well for the domain from the lower bound to a certain point. After that point, the curvature of the function under consideration grows and EMPR approximant can not confront this increment. On the other hand, constant FFI-Piecewise EMPR approximant can overcome this situation easily. It can be verified that the green surface overlaps the exact surface which is the red one in the vicinity of the upper bounds of the independent variables.

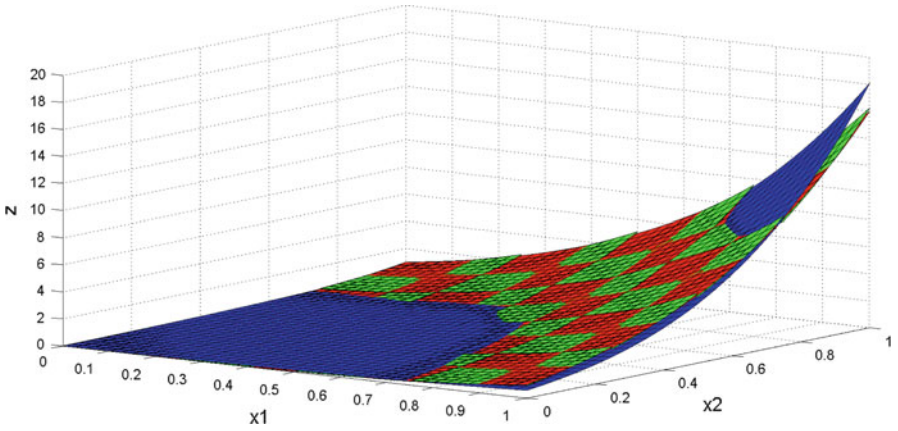

Fig. 1 Exact function f_8 (*red*), constant standard EMPR approximant (*blue*), constant FFI-Piecewise EMPR approximant with 4 subintervals (*green*) (Color figure online)

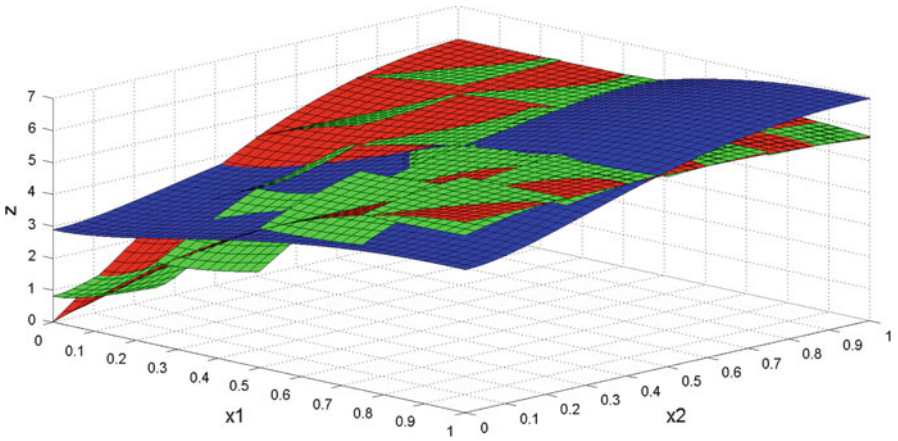

Fig. 2 Exact function f_9 (*red*), constant standard EMPR approximant (*blue*), constant FFI-Piecewise EMPR approximant with 4 subintervals (*green*) (Color figure online)

In Fig. 2, it is obvious that the green surfaces obtained by applying FFI-Piecewise EMPR to the relevant function express the exact surface more accurately than standard EMPR does for almost all over the domain. This shows the efficiency of the FFI-Piecewise EMPR method one more time.

6 Conclusion

The EMPR method has a finite expansion composed of a number of less-variate functions and is used to approximately represent a given multivariate function in terms of the components of its expansion. This expansion has also support functions in its structure which makes EMPR different from HDMR. These support functions also

allow the method to work efficiently for representing multivariate functions having different types of natures.

In this work, we apply the Fluctuation Free Integration algorithm to the steps of the EMPR method to evaluate the multiple integrals appearing in the method numerically, not analytically which reduces the mathematical and computational complexity of EMPR. This results in a new method named as FFI-EMPR. In addition, we also try to increase the performance of this FFI-EMPR method and a new piecewise based EMPR philosophy is proposed to determine better approximations to the given multivariate functions. This method is called FFI-Piecewise EMPR. It splits the whole problem geometry into a number of subgeometries and obtains an approximant for each subgeometry. This approach constitutes a piecewise structure at the end.

We used constant and univariate FFI-Piecewise EMPR approximants in representing multivariate functions. Then, we compare the obtained approximants with the standard EMPR approximants and the original function. The numerical results given through tables and figures in the numerical implementations section show us that the FFI-Piecewise EMPR approximants are more successfull than standard EMPR in representing multivariate functions. Besides, the relative error values also show that the approximations are quite sufficient in representing functions when compared with the original ones. In detail, we can conclude that even the constant FFI-Piecewise EMPR approximant with 4 subgeometries works better than the univariate EMPR approximant in the representation process of multivariate functions. Of course, as the number of subintervals increases the performance of FFI-Piecewise EMPR gets better. But, not to increase the computational complexity one can easily stop at 4 in deciding the total number of subgeometries for our proposed method. This brings us a great opportunity in modelling, that is, we can make a better approximation without increasing the computational complexity of the algorithm.

In addition, the proposed method of this work successfully represents the multivarite functions even they have high curvature. It is known that the representation of multivariate functions having high curvature is quite hard by using standard methods in numerical analysis. This success comes from splitting the given geometry through the FFI-Piecewise EMPR. This situaton can be easily seen by examining the figures of the previous section.

Of course, there are also some other ways to obtain a representation for a multivariate function through EMPR based methods. One way may be the support function optimization process which needs a detailed approach in solving nonlinear equation systems appearing in its structure and it is totally out of scope of this work. Hence, this fact is left as a future work.

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References

1. B. Tunga, M. Demiralp, The influence of the support functions on the quality of Enhanced Multivariance Product Representation. J. Math. Chem. **48**, 827–840 (2010)

2. M. Demiralp, Dominating the constancy in Enhanced Multivariance Product Representation (EMPR) via support function selection. In *Proceedings of the 2nd International Conference on Applied Informatics and Computing Theory (AICT'11)*, 26–28 September, Keynote Speech, Prague, Czech Republic, pp. 11–12 (2011)
3. B. Kalay, M. Demiralp, Affine transformational Enhanced Multivariance Product Representation (ATEMPR) and its relation to rational approximants. In *Proceedings for the 1st IEEEAM Conference on Applied Computer Science (ACS)*, pp. 336–340 (2010)
4. I.M. Sobol, Sensitivity estimates for nonlinear mathematical models. Math. Model. Comput. Exp. (MMCE) **1**(4), 407–414 (1993)
5. M. Demiralp, High dimensional model representation and its application varieties. Math. Res. **9**, 146–159 (2003)
6. G. Li, H. Rabitz, General formulation of HDMR component functions with independent and correlated variables. J. Math. Chem. **50**, 99–130 (2012)
7. T. Ziehn, A.S. Tomlin, A global sensitivity study of sulfur chemistry in a premixed methane flame model using HDMR. Int. J. Chem. Kinet. **40**, 742–753 (2008)
8. J. Sridharan, T. Chen, Modeling multiple input switching of CMOS gates in DSM technology using HDMR. Proc. Des. Autom. Test Eur. **1**, 624–629 (2006)
9. B.N. Rao, R. Chowdhury, Probabilistic analysis using high dimensional model representation and fast fourier transform. Int. J. Comput. Methods Eng. Sci. Mech. **9**, 342–357 (2008)
10. A. Okan, N.A. Baykara, M. Demiralp, Weight optimization in Enhanced Multivariance Product Representation (EMPR) method. International conference of numerical analysis and applied mathematics. AIP Conf. Proc. **1281**, 1935–1938 (2010)
11. M. Demiralp, A new fluctuation expansion based method for the univariate numerical integration under Gaussian weights. In *WSEAS 8-th International Conference on Applied Mathematics*, Tenerife, Spain, pp. 68–73 (2005)
12. M. Demiralp, Convergence issues in the Gaussian weighted multidimensional fluctuation expansion for the univariate numerical integration. WSEAS Trans. Math. **4**, 486–492 (2005)
13. M. Demiralp, Fluctuationlessness theorem to approximate univariate functions matrix representations. WSEAS Trans. Math. **8**, 258–267 (2009)
14. M. Demiralp, Data production for a multivariate function on an orthogonal hyperprismatic grid via fuctuation free matrix representation: completely filled grid case. IJEECE **1**, 61–76 (2010)
15. A. Langer, S. Osher, C. Schnlieb, Bregmanized domain decomposition for image restoration. J. Sci. Comput. **54**, 549–576 (2013)
16. J. Geer, N.S. Banerjee, Exponentially accurate approximations to piece-wise smooth periodic functions. J. Sci. Comput. **12**(3), 253–287 (1997)
17. M.A. Tunga, M. Demiralp, Multivariate data modelling through piecewise generalized HDMR method. J. Math. Chem. **5**, 1711–1726 (2012)
18. B. Tunga, M. Demiralp, Support function influences on the univariance of the enhanced multivariance product representation. In *Seventh International Conference of Computational Methods in Sciences and Engineering*, Rhodes, Greece, 2009
19. B. Tunga, M. Demiralp, Fluctuation free multivariate integration based logarithmic HDMR in multivariate function representation. J. Math. Chem. **49**, 894–909 (2011)